



**You have downloaded a document from  
RE-BUŚ  
repository of the University of Silesia in Katowice**

**Title:** Enhancing Pseudo-Telepathy in the Magic Square Game

**Author:** Łukasz Pawela, Piotr Gawron, Zbigniew Puchała, Jan Ślădkowski

**Citation style:** Pawela Łukasz, Gawron Piotr, Puchała Zbigniew, Ślădkowski Jan. (2013). Enhancing Pseudo-Telepathy in the Magic Square Game. "PLoS ONE" (2013, Vol. 8, no. 6, art. no. 64694), doi 10.1371/journal.pone.0064694



Uznanie autorstwa - Licencja ta pozwala na kopiowanie, zmienianie, rozprowadzanie, przedstawianie i wykonywanie utworu jedynie pod warunkiem oznaczenia autorstwa.



UNIwersYTET ŚLĄSKI  
W KATOWICACH



Biblioteka  
Uniwersytetu Ślăskiego



Ministerstwo Nauki  
i Szkolnictwa Wyższego

# Enhancing Pseudo-Telepathy in the Magic Square Game

Łukasz Paweła<sup>1\*</sup>, Piotr Gawron<sup>1</sup>, Zbigniew Puchała<sup>1,2</sup>, Jan Ślaskowski<sup>3</sup>

**1** Institute of Theoretical and Applied Informatics, Polish Academy of Sciences, Gliwice, Poland, **2** Institute of Physics, Jagiellonian University, Kraków, Poland, **3** Institute of Physics, University of Silesia, Katowice, Poland

## Abstract

We study the possibility of reversing an action of a quantum channel. Our principal objective is to find a specific channel that reverses as accurately as possible an action of a given quantum channel. To achieve this goal we use semidefinite programming. We show the benefits of our method using the quantum pseudo-telepathy Magic Square game with noise. Our strategy is to move the pseudo-telepathy region to higher noise values. We show that it is possible to reverse the action of a noise channel using semidefinite programming.

**Citation:** Paweła Ł, Gawron P, Puchała Z, Ślaskowski J (2013) Enhancing Pseudo-Telepathy in the Magic Square Game. PLoS ONE 8(6): e64694. doi:10.1371/journal.pone.0064694

**Editor:** Dennis Salahub, University of Calgary, Canada

**Received:** March 8, 2013; **Accepted:** April 17, 2013; **Published:** June 6, 2013

**Copyright:** © 2013 Paweła et al. This is an open-access article distributed under the terms of the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original author and source are credited.

**Funding:** This work was supported by the Polish National Science Centre: Ł. Paweła by the grant number N N516 481840, J. Ślaskowski by the grant number DEC-2011/01/B/ST6/07197. Work was also supported by the Polish Ministry of Science and Higher Education: P. Gawron under the project number IP2011 014071, Z. Puchała under the project number IP2011 044271. The funders had no role in study design, data collection and analysis, decision to publish, or preparation of the manuscript.

**Competing Interests:** The authors have declared that no competing interests exist.

\* E-mail: lpawela@iitis.pl

## Introduction

Quantum game theory is an interdisciplinary field that combines game theory and quantum information. It lies at the crossroads of physics, quantum information processing, computer and natural sciences. Various quantizations of games were presented by different authors [1–5].

Quantum pseudo-telepathy games [6] form a subclass of quantum games. A game belongs to the pseudo-telepathy class providing that there are no winning strategies for classical players, but a winning strategy can be found if the players share a sufficient amount of entanglement. In these games quantum players can accomplish tasks that are unfeasible for their classical counterparts. It has been shown [7] that noise in a quantum channel can decrease the probability of winning the Magic Square game even below the classical threshold.

Noise is an unavoidable ingredient of a quantum system. Therefore its thorough investigation is a fundamental issue in quantum information processing. Quantum game theory has several potential applications (e.g. quantum auctions [8]) that may be hindered by noise effects. Our previous investigation of quantum noise effects in quantum games [7,9,10], and quantum algorithms performance [11] revealed several interesting issues that act as an incentive of the present work. The tools developed in this work can be used to analyse the behaviour of quantum channels in other settings.

## Motivation

The motivation to study the Magic Square game and pseudo-telepathy games in general is that their physical implementation could provide convincing, even to a layperson, demonstration that the physical world is not local realistic. By *local* we mean that no action performed at some location X can have an effect on some remote location Y in a time shorter than that required by light to

travel from X to Y. *Realistic* means that a measurement can only reveal elements of reality that are already present in the system [6].

Given a pseudo-telepathy game, one can implement a quantum winning strategy for this game [6]. In an ideal case, the experiment should involve a significant number of rounds of the game. The experiment should be continued until either the players lose a single round or the players win such a great number of rounds, that it would be nearly impossible if they were using a classical strategy.

In the particular case of the magic square game the classical strategy allows the players to achieve the success rate no larger than  $\frac{8}{9}$ . In theory, the success rate of the quantum strategy is equal to one. But any physical implementations of a quantum protocol cannot be perfect because it is subject to noise.

In particular, the players, Alice and Bob, must be so far away from each other that the time between the question and their respective answers is shorter than the time required by light to travel between their locations. This set-up involves sending parts of an entangled quantum state to two remote locations. Sending qubits through a channel will inevitably add noise to the system. Our aim is to counteract this noise. In this paper we focus on the destructive aspects of the process of transmission of a qubit through a noisy separable quantum channel and introduce a scheme that allows the partial reversion of the channel action. This reversal gives rise to the players' success rate above the classical limit of  $\frac{8}{9}$  for some parameters of noisy channels. Our scheme for reversing an action of a noisy channel may prove valuable in future experimental set-ups of such games.

## Magic square game

The magic square is a  $3 \times 3$  matrix filled with numbers 0 or 1 so that the sum of entries in each row is even and the sum of entries in each column is odd. Although such a matrix cannot exist (see Table 1) one can consider the following game.

**Table 1.** An illustrative filling of the magic square with numbers 0 and 1.

1	1	0	
1	0	1	
1	0	?	

The question mark shows that it is not possible to put a number in the last field and satisfy both conditions of the game.  
doi:10.1371/journal.pone.0064694.t001

The game setup is as follows. There are two players: Alice and Bob. Alice is given a row, Bob is given a column. Alice has to give the entries for a row and Bob has to give entries for a column so that the parity conditions are met. Winning condition is that the players' entries at the intersection must agree. Alice and Bob can prepare a strategy but they are not allowed to communicate during the game.

There exists a (classical) strategy that guarantees the winning probability of  $\frac{8}{9}$ . If the parties are allowed to share a quantum state they can achieve probability of success equal to one [6].

In the quantum version of this game [12,13] Alice and Bob are allowed to share an entangled quantum state. The winning strategy is following. Alice and Bob share an entangled state:

$$|\psi\rangle = \frac{1}{2}(|0011\rangle + |1100\rangle - |0110\rangle - |1001\rangle) \quad (1)$$

and apply local unitary operators forming operator  $A_i \otimes B_j$ , where

$$\begin{aligned} \bullet A_1 &= \frac{1}{\sqrt{2}} \begin{pmatrix} i & 0 & 0 & 1 \\ 0 & -i & 1 & 0 \\ 0 & i & 1 & 0 \\ 1 & 0 & 0 & i \end{pmatrix}, \\ \bullet A_2 &= \frac{1}{2} \begin{pmatrix} i & 1 & 1 & i \\ -i & 1 & -1 & i \\ i & 1 & -1 & -i \\ -i & 1 & 1 & -i \end{pmatrix}, \\ \bullet A_3 &= \frac{1}{2} \begin{pmatrix} -1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} \bullet B_1 &= \frac{1}{2} \begin{pmatrix} i & -i & 1 & 1 \\ -i & -i & 1 & -1 \\ 1 & 1 & -i & i \\ -i & i & 1 & 1 \end{pmatrix}, \\ \bullet B_2 &= \frac{1}{2} \begin{pmatrix} -1 & i & 1 & i \\ 1 & i & 1 & -i \\ 1 & -i & 1 & i \\ -1 & -i & 1 & -i \end{pmatrix}, \\ \bullet B_3 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix}. \end{aligned}$$

Indices  $i$  and  $j$  label rows and columns of the magic square. The state of this scheme before measurement is

$$\rho_f = (A_i \otimes B_j) |\psi\rangle \langle \psi| (A_i^\dagger \otimes B_j^\dagger). \quad (2)$$

The final step of the game consists of the measurement in the computational basis.

In [7], the situation where the initial state  $|\psi\rangle$  is corrupted by the noise was investigated. Therefore, Eq. 2 is transformed into

$$\rho_f = (A_i \otimes B_j) \Phi_\alpha(|\psi\rangle \langle \psi|) (A_i^\dagger \otimes B_j^\dagger), \quad (3)$$

where  $\Phi_\alpha$  denotes one-parameter family of noisy quantum channels.

In such a case it is justified to inquire what is the mean probability of Alice and Bob's success given the amount of noise introduced by channel  $\Phi_\alpha$ . The mean probability  $p(\alpha)$  of measuring the outcome yielding success in the state  $\rho_f$  is given by

$$p(\alpha) = \frac{1}{9} \sum_{i,j=1}^3 \sum_{\zeta \in \mathcal{S}_{ij}} \text{tr} \rho_f |\zeta\rangle \langle \zeta|, \quad (4)$$

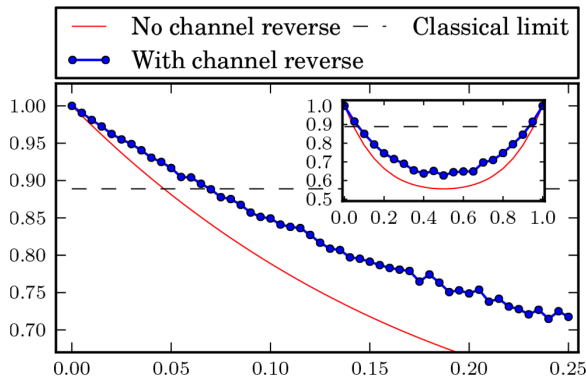
where  $\mathcal{S}_{ij}$  is the set of right answers for the column and row  $ij$  (Table 2). The mean is taken over all pairs  $(i,j)$ .

A winning strategy exists for noiseless channels. In the case of noisy channel, the same strategy gives a higher probability of winning than in the classical case for low noise channels [7]. The objective of this work is to find local channels that partially reverse the action of the noise and therefore extends the pseudo-telepathy

**Table 2.** Sets  $\mathcal{S}_{ij}$  – plus sign (+) indicates that the given element belongs to the set, minus (−) sign indicates that the element does not belong to the set.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\mathcal{S}_{11}$	+	+	−	−	+	+	−	−	−	−	+	+	−	−	+	+
$\mathcal{S}_{12}$	+	+	−	−	−	−	+	+	+	+	−	−	−	−	+	+
$\mathcal{S}_{13}$	+	+	−	−	−	−	+	+	−	−	+	+	+	+	−	−
$\mathcal{S}_{21}$	+	−	+	−	+	−	+	−	−	+	−	+	−	+	−	+
$\mathcal{S}_{22}$	+	−	+	−	−	+	−	+	+	−	+	−	−	+	−	+
$\mathcal{S}_{23}$	+	−	+	−	−	+	−	+	−	+	−	+	+	−	+	−
$\mathcal{S}_{31}$	−	+	+	−	−	+	+	−	+	−	−	+	+	−	−	+
$\mathcal{S}_{32}$	−	+	+	−	+	−	−	+	−	+	+	−	+	−	−	+
$\mathcal{S}_{33}$	−	+	+	−	+	−	−	+	+	−	−	+	−	+	+	−

doi:10.1371/journal.pone.0064694.t002



**Figure 1. Phase flip channel.** Probability of winning the pseudo-telepathy game with and without the use of our approach as a function of the noise parameter  $\alpha$  for the phase flip channel. The inset shows the probability of winning for  $\alpha \in [0; 1]$ . doi:10.1371/journal.pone.0064694.g001

to channels with higher noise. In order to achieve this, Eq. 3 is transformed into

$$\rho_f = (A_i \otimes B_j) \Psi_\alpha(|\psi\rangle\langle\psi|) (A_i^\dagger \otimes B_j^\dagger), \quad (5)$$

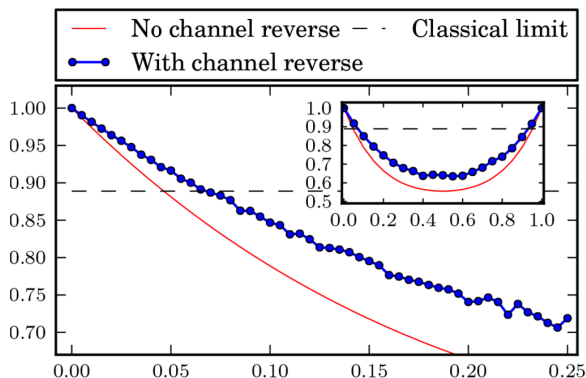
where  $\Psi_\alpha$  denotes local channel with respect to Alice and Bob's subsystems that allows to raise their probability of winning  $p(\alpha)$ . In order to achieve that a series of semi-definite optimization programs has to be numerically solved.

### Quantum channels

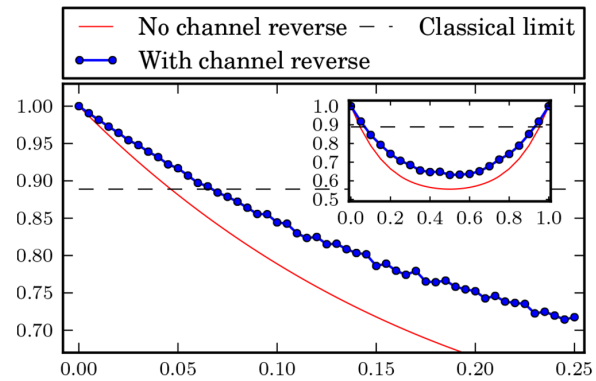
In the most general case, the evolution of a quantum system can be described using the notion of a *quantum channel* [14–16]. A quantum channel is a mapping acting on density operators  $\rho \in D(\mathcal{H})$ , i.e., operators where  $\rho \geq 0$  and  $\text{tr}(\rho) = 1$  on a Hilbert space  $\mathcal{H}_1$  and transforming them into operators on another Hilbert space  $\mathcal{H}_2$ . Thus

$$\Phi : L(\mathcal{H}_1) \rightarrow L(\mathcal{H}_2), \quad (6)$$

where  $L(\mathcal{H}_i)$  denotes the set of linear operators on  $\mathcal{H}_i$ . To form a



**Figure 2. Bit flip channel.** Probability of winning the pseudo-telepathy game with and without the use of our approach as a function of the noise parameter  $\alpha$  for the bit flip channel. The inset shows the probability of winning for  $\alpha \in [0; 1]$ . doi:10.1371/journal.pone.0064694.g002



**Figure 3. Bit phase flip channel.** Probability of winning the pseudo-telepathy game with and without the use of our approach as a function of the noise parameter  $\alpha$  for the bit phase flip channel. The inset shows the probability of winning for  $\alpha \in [0; 1]$ . doi:10.1371/journal.pone.0064694.g003

proper quantum channel, the mapping  $\Phi$  must satisfy the following restrictions:

1.  $\Phi$  must be *trace-preserving*, that is  $\text{tr}(\Phi(\rho)) = \text{tr}(\rho)$ ,
2.  $\Phi$  must be *completely positive*, that is  $\Phi \otimes \mathbb{I}_{L(\mathcal{H}_3)}$  is a positive mapping, thus

$$(\Phi \otimes \mathbb{I}_{L(\mathcal{H}_3)})(\rho) \in D(\mathcal{H}_2 \otimes \mathcal{H}_3), \quad (7)$$

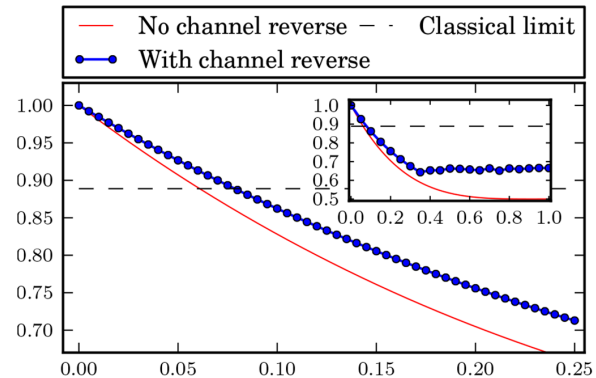
for every choice of  $\rho \in D(\mathcal{H}_1 \otimes \mathcal{H}_3)$  and every choice of finite-dimensional Hilbert space  $\mathcal{H}_3$ , where  $\mathbb{I}_{L(\mathcal{H}_3)}$  is an identity channel on the space  $L(\mathcal{H}_3)$ .

The notion of a *product* quantum channel is introduced as follows [17]. For any choice of quantum channels that satisfy

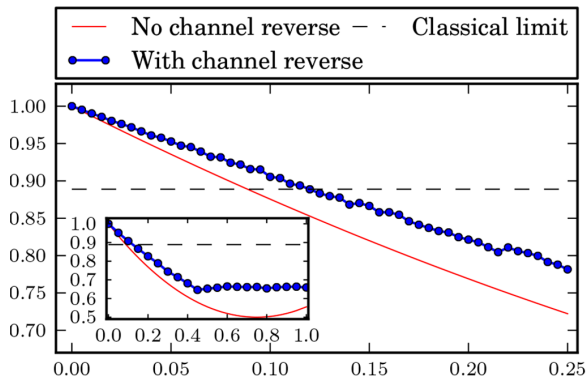
$$\Phi_1 : L(\mathcal{H}_1^1) \rightarrow L(\mathcal{H}_2^1), \dots, \Phi_N : L(\mathcal{H}_1^N) \rightarrow L(\mathcal{H}_2^N), \quad (8)$$

we define a linear mapping

$$\Phi_1 \otimes \dots \otimes \Phi_N : L(\mathcal{H}_1^1 \otimes \dots \otimes \mathcal{H}_1^N) \rightarrow L(\mathcal{H}_2^1 \otimes \dots \otimes \mathcal{H}_2^N), \quad (9)$$



**Figure 4. Depolarising channel.** Probability of winning the pseudo-telepathy game with and without the use of our approach as a function of the noise parameter  $\alpha$  for the depolarising channel. The inset shows the probability of winning for  $\alpha \in [0; 1]$ . doi:10.1371/journal.pone.0064694.g004



**Figure 5. Amplitude damping channel.** Probability of winning the pseudo-telepathy game with and without the use of our approach as a function of the noise parameter  $\alpha$  for the amplitude damping channel. The inset shows the probability of winning for  $\alpha \in [0; 1]$ . doi:10.1371/journal.pone.0064694.g005

to be the unique mapping that satisfies the equation

$$(\Phi_1 \otimes \dots \otimes \Phi_N)(A_1 \otimes \dots \otimes A_N) = \Phi_1(A_1) \otimes \dots \otimes \Phi_N(A_N), \quad (10)$$

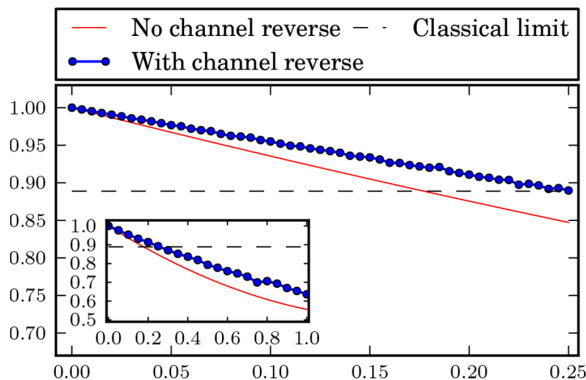
for all operators  $A_1 \in L(\mathcal{H}_1), \dots, A_N \in L(\mathcal{H}_1^N)$ .

Many different representations of quantum channels can be chosen, depending on the application. Among these are the Jamiolkowski representation, the Kraus representation and the Stinespring representation. These three representations will be used throughout this paper.

The Jamiolkowski representation of a quantum channel  $\Phi$  is given by

$$\mathcal{J}(\Phi) = \sum_{a,b} \Phi(E_{a,b}) \otimes E_{a,b}, \quad (11)$$

where  $E_{a,b} \in L(\mathcal{H}_1)$  are operators with all entries equal to zero, except the entry  $a,b$  equal to one. From this definition, it is straightforward to observe that  $\mathcal{J}(\Phi) \in L(\mathcal{H}_2 \otimes \mathcal{H}_1)$ . By the Choi's [14] theorem a channel is completely positive if and only if  $\mathcal{J}(\Phi) \geq 0$ . It is trace-preserving if and only if



**Figure 6. Phase damping channel.** Probability of winning the pseudo-telepathy game with and without the use of our approach as a function of the noise parameter  $\alpha$  for the phase damping channel. The inset shows the probability of winning for  $\alpha \in [0; 1]$ . doi:10.1371/journal.pone.0064694.g006

$$\text{tr}_{\mathcal{H}_2}(\mathcal{J}(\Phi)) = \mathbb{I}_{\mathcal{H}_1}. \quad (12)$$

Finally, the action of a quantum channel in the Jamiolkowski representation is given by

$$\Phi(\rho) = \text{tr}_{\mathcal{H}_1}(\mathcal{J}(\Phi)(\mathbb{I}_{\mathcal{H}_2} \otimes \rho^T)). \quad (13)$$

The Kraus representation of a quantum channel is given by a set of operators  $E_k \in L(\mathcal{H}_1, \mathcal{H}_2)$ . The action of quantum channel  $\Phi$  is given by:

$$\Phi(\rho) = \sum_k E_k \rho E_k^\dagger. \quad (14)$$

This form ensures that the quantum channel is completely positive. For it to be also trace-preserving we need to impose the following constraint on the Kraus operators

$$\sum_k E_k^\dagger E_k = \mathbb{I}_{\mathcal{H}_1}. \quad (15)$$

Finally, given a mapping  $\Phi: L(\mathcal{H}_1) \rightarrow L(\mathcal{H}_2)$  let us take another Hilbert space  $\mathcal{H}_3$  such that  $\dim(\mathcal{H}_3) = \text{rank}(\mathcal{J}(\Phi))$  and a linear isometry  $A \in U(\mathcal{H}_1, \mathcal{H}_2 \otimes \mathcal{H}_3)$ . The action of a quantum channel is given by

$$\Phi(\rho) = \text{tr}_{\mathcal{H}_3}(A \rho A^\dagger). \quad (16)$$

This representation is called the Stinespring representation of  $\Phi$ . For further discussion of quantum channels see e.g. [15] or [17].

### Quantum noise

In the literature, several one-parameter families of qubit noisy channels are discussed [15]. For all the families of channels listed below the parameter  $\alpha \in [0, 1]$  represents the amount of noise introduced by the channel. The symbols  $\sigma_x, \sigma_y, \sigma_z$  denote Pauli operators. The Kraus operators for typical noisy channels are for

- • depolarising channel:

$$\left\{ \sqrt{1 - \frac{3\alpha}{4}} \mathbb{I}, \sqrt{\frac{\alpha}{4}} \sigma_x, \sqrt{\frac{\alpha}{4}} \sigma_y, \sqrt{\frac{\alpha}{4}} \sigma_z \right\},$$

- • amplitude damping:  $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\alpha} \end{pmatrix}, \begin{pmatrix} 0 & \sqrt{\alpha} \\ 0 & 0 \end{pmatrix} \right\},$

- • phase damping:  $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\alpha} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\alpha} \end{pmatrix} \right\},$

- • phase flip:  $\left\{ \sqrt{1-\alpha} \mathbb{I}, \sqrt{\alpha} \sigma_z \right\},$

- • bit flip:  $\left\{ \sqrt{1-\alpha} \mathbb{I}, \sqrt{\alpha} \sigma_x \right\},$

```

input : A random Jamiołkowski matrix,  $Y$ , the matrices  $W$ ,  $\sigma$ ,  $\tau$  and the number of runs  $n$ 
output: Optimized values of the parts of the product channel

calculate  $M$ ;
for  $i \leftarrow 1$  to  $n$  do
     $Z = \text{SolveSDP}(\text{Eqn. (24)}, Y, M)$ ;
     $Y = \text{SolveSDP}(\text{Eqn. (25)}, Z, M)$ ;
end
return  $W(Y \otimes Z)W$ 

```

**Figure 7. SDP optimization algorithm.** SDP optimization of a product channel with a random Jamiołkowski matrix [20] as input. doi:10.1371/journal.pone.0064694.g007

- bit-phase flip:  $\{\sqrt{1-\alpha}\mathbb{I}, \sqrt{\alpha}\sigma_y\}$ .

In order to apply noise operators to multiple qubits we form new set of operators acting on a larger Hilbert space.

Given a set of  $n$  one-qubit Kraus operators  $\{e_k\}_{k=1}^n$  it is possible to construct new set of  $n^N$  operators  $\{E_k\}_{k=1}^{n^N}$  that act on a Hilbert space of dimension  $2^N$  by taking Cartesian product of one-qubit Kraus operators in the following way

$$\{E_k\}_{k=1}^{n^N} = \{e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_N}\}_{i_1, i_2, \dots, i_N=1}^n. \quad (17)$$

Application of the above to Kraus operators listed before gives one-parameter families of local noisy channels. This form will be used in further investigations.

## Results and Discussion

We propose the following scheme for reversing an action of a channel using semidefinite programming (SDP). In our case, the most useful formulation of a semidefinite program is as follows (after Watrous [17]).

A semidefinite program is a triple  $(\Phi, A, B)$  where  $\Phi: L(\mathcal{H}_1) \rightarrow L(\mathcal{H}_2)$  is a Hermiticity-preserving map and  $A \in L(\mathcal{H}_1)$  and  $B \in L(\mathcal{H}_2)$  are Hermitian operators for some choice of Hilbert spaces  $\mathcal{H}_1$  and  $\mathcal{H}_2$ . Two optimization problems are associated with the triple  $(\Phi, A, B)$ , the primal and dual problems. We will focus our attention on the primal problem, which has the form:

$$\text{maximize : } \text{tr}(AX),$$

$$\text{subject to : } \Phi(X) = B,$$

$$X \geq 0.$$

In the case of the pseudo-telepathy game, it seems appropriate to look for a channel in a product form. This is due to the fact, that Alice and Bob are separated and each of them must apply a local channel. To model this situation, let us consider the Jamiołkowski representations of Alice's and Bob's channels, denoted  $Y$  and  $Z$  respectively. The resulting channel is given by

$$T = W(Y \otimes Z)W, \quad (18)$$

where  $W$  is an operator defined as follows

$$W = \mathbb{I}_{\mathcal{H}_2^A} \otimes U \otimes \mathbb{I}_{\mathcal{H}_1^B}, \quad (19)$$

where  $U \in L(\mathcal{H}_1^A \otimes \mathcal{H}_2^B, \mathcal{H}_2^B \otimes \mathcal{H}_1^A)$  is the swap operation of subsystems  $\mathcal{H}_1^A$  and  $\mathcal{H}_2^B$ , defined as

$$U = \sum_{i,j} |f_j e_i\rangle \langle e_i f_j|, \quad (20)$$

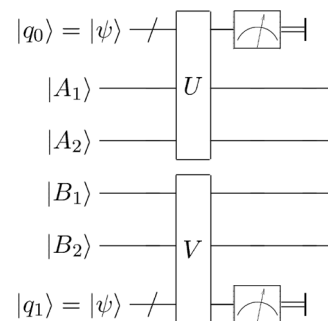
for  $e_i, f_j$  being elements of orthonormal bases of  $\mathcal{H}_1^A$  and  $\mathcal{H}_2^B$  respectively.

Next, let us denote by  $\Psi_N$  the noise channel and we put  $\sigma = |\psi\rangle\langle\psi|$ . For simplicity of further calculations, let us write  $\tau = \Psi_N(\sigma)$  and  $T = \mathcal{J}(\Psi_N)$ . Consider the following maximization criterion problem

$$\text{maximize : } \text{tr}(\text{tr}_2(W(Y \otimes Z)W(\mathbb{I} \otimes \tau^T))\sigma), \quad (21)$$

which means we aim to find a channel that reverses the action of the noise channel as accurately as possible. Unfortunately, a maximization criterion in this form does not yield an SDP problem. To formulate this problem as an SDP, we first conduct some simple calculations that allow us to rewrite the maximization condition (21) as

$$\text{maximize : } \text{tr}((Y \otimes Z)M), \quad (22)$$



**Figure 8. Quantum circuit.** A quantum circuit showing the implementation of our scheme.  $A_i$  and  $B_i$  denote Alice's and Bob's qubits.  $q_0$  and  $q_1$  are the ancillary qubits they need to add. doi:10.1371/journal.pone.0064694.g008



$$\text{subject to : } M = W(\sigma \otimes \tau)W. \quad (23)$$

Considering the value of  $Y$  to be fixed and using the equation  $\text{tr}((\mathbb{I} \otimes A)B) = \text{tr}(A \text{tr}_1(B))$ , allows us to write the following SDP

$$\begin{aligned} &\text{maximize : } \text{tr}(Z \text{tr}_{\mathcal{H}_2^B, \mathcal{H}_1^B}(M(Y \otimes \mathbb{I}_{\mathcal{H}_2^B \otimes \mathcal{H}_1^B}))), \\ &\text{subject to : } \text{tr}_{\mathcal{H}_2^B}(Z) = \mathbb{I}_{\mathcal{H}_1^B}, \end{aligned} \quad (24)$$

$$Z \geq 0.$$

Fixing the value of  $Z$  and following a similar calculation give the following SDP problem

$$\begin{aligned} &\text{maximize : } \text{tr}(Y \text{tr}_{\mathcal{H}_2^A, \mathcal{H}_1^A}((\mathbb{I}_{\mathcal{H}_2^A \otimes \mathcal{H}_1^A} \otimes Z)M)), \\ &\text{subject to : } \text{tr}_{\mathcal{H}_2^A}(Y) = \mathbb{I}_{\mathcal{H}_1^A}, \end{aligned} \quad (25)$$

$$Y \geq 0.$$

Now, we use the following algorithm to find an optimal channel. The algorithm in each iteration optimizes only a single part of the product channel. This algorithm was implemented using the SDPLR library [18,19].

## Analysis

The numerical results are gathered in form of plots at the end of the paper. Figs. 1, 2, 3, 4, 5, 6 show the results of the optimization scheme shown in Fig. 7. The application of the SDP allowed us to achieve greater winning probability for all types of noisy channels.

In the case of the flip channels the obtained results are depicted in Figs. 1, 2 and 3. These plots show that it is possible to reverse the action of the noise channel for all values of the noise parameter  $\alpha$ . Hence, we are able to observe quantum pseudo-telepathy for higher noise channels. Furthermore, the use of our optimization method results in a plot of probability of winning as a function of the noise parameter  $\alpha$  which has a shape similar to the case when we do not try to reverse the action of a channel.

Next, we move to the depolarising channel. The results obtained in this case are shown in Fig. 4. Likewise, in this case our method has also allowed us to achieve pseudo-telepathy for higher values of the noise parameter  $\alpha$ . The details are depicted in the inset in Fig. 4. Additionally, for values of the noise parameter

$\alpha \geq 0.45$  the probability of winning the game stabilizes around 0.65, opposed to the case with no channel action inverse, where it decreases to 0.5. Hence, we are able to retrieve some information in the case of high noise, local depolarising channels acting on many qubits.

Finally, we switch to the damping channels. Numerical results for this case are depicted in Figs. 5 and 6. Moreover, in this case we are able to reverse the action of a noise channel and broaden the pseudo-telepathy region. In the case of high values of the noise parameter  $\alpha$ , results for the amplitude damping channel resemble those obtained for depolarising channel, as the probability of winning stabilizes around 0.65 for  $\alpha \geq 0.45$  instead of decreasing to approximately 0.5.

## Conclusions

The principal result of this paper is a methodology of partial denoising with the usage of local quantum channels. The presented tool can be used in the cases in which

- the parameters of the noise are accessible,
- the noisy channel is separable and acts independently on each qubit,
- the entangled quantum state the parties use is known in advance,
- the parties have access to quantum computers but
- are not allowed to communicate.

We have proposed a method to reverse an action of a quantum channel using semidefinite programming. The method allows us to find a product channel which partially reverses a given channel. We use the following scheme to achieve this goal. First, we fix all parts of the product, except for one, which is being optimized. After the SDP optimization, we move on to optimize the next part of the product channel, using the value obtained in the earlier step. We repeat this for all parts of the product channel. We run the process a great number of times to obtain a converging solution.

Obtained channel may be implemented on a real physical system using the Stinespring representation. An example of the quantum circuit implementing this scheme is shown in Fig. 8. Alice and Bob each add ancillary qubits to their original ones. Then they apply a unitary operator on their respective systems. Finally, they perform a measurement on the ancillary qubits, leaving their starting qubits in a less noisy state.

As an example of usage of this optimization scheme we present the quantum pseudo-telepathy magic square game. We obtained results showing an improvement in the players' success rate in the game. Specifically, we were able to broaden the range of the noise parameter  $\alpha$  for which the quantum effect occurs.

## Author Contributions

Conceived and designed the experiments: PG. Performed the experiments: LP. Analyzed the data: LP ZP. Contributed reagents/materials/analysis tools: PG ZP. Wrote the paper: LP PG ZP JS.

## References

1. Eisert J, Wilkens M, Lewenstein M (1999) Quantum games and quantum strategies. *Physical Review Letters* 83: 3077–3080.
2. Piotrowski EW, Sładkowski J (2003) An invitation to quantum game theory. *International Journal of Theoretical Physics* 42: 1089–1099.
3. Flitney AP, Abbott D (2002) Quantum version of the monty hall problem. *Phys Rev A* 65: 062318.
4. Frąckiewicz P (2012) Quantum information approach to normal representation of extensive games. *International Journal of Quantum Information* 10.
5. Schmidt AGM, da Silva L (2012) Quantum russian roulette. *Physica A: Statistical Mechanics and its Applications*.
6. Brassard G, Broadbent A, Tapp A (2005) Quantum pseudo-telepathy. *Foundations of Physics* 35: 1877–1907.
7. Gawron P, Miszcak JA, Sładkowski J (2008) Noise effects in quantum magic squares game. *International Journal of Quantum Information* 06: 667.
8. Piotrowski E, Sładkowski J (2008) Quantum auctions: Facts and myths. *Physica A: Statistical Mechanics and its Applications* 387: 3949–3953.

9. Gawron P (2010) Noisy quantum monty hall game. *Fluctuation and Noise Letters* 9: 9–18.
10. Miszczak J, Gawron P, Puchala Z (2012) Qubit flip game on a heisenberg spin chain. *Quantum Information Processing* 11: 1571–1583.
11. Gawron P, Klamka J, Winiarczyk R (2012) Noise effects in the quantum search algorithm from the viewpoint of computational complexity. *International Journal of Applied Mathematics and Computer Science* 22: 493–499.
12. Mermin N (1990) Simple unified form for the major no-hidden-variables theorems. *Physical Review Letters* 65: 3373–3376.
13. Aravind PK (2004) Quantum mysteries revisited again. *American Journal of Physics* 72: 1303.
14. Bengtsson I, Życzkowski K (2006) *Geometry of Quantum States. An Introduction to Quantum Entanglement*. Cambridge University Press.
15. Nielsen MA, Chuang IL (2000) *Quantum Computation and Quantum Information*. Cambridge University Press.
16. Puchala Z, Miszczak J, Gawron P, Gardas B (2011) Experimentally feasible measures of distance between quantum operations. *Quantum Information Processing* 10: 1–12.
17. Watrous J (2011). *Theory of quantum information*, Lecture notes CS 766/QJC 820. published online. URL <http://www.cs.uwaterloo.ca/watrous/CS766/LectureNotes/all.pdf>. <http://www.cs.uwaterloo.ca/watrous/CS766/>.
18. Burer S, Monteiro RDC (2003) A nonlinear programming algorithm for solving semidefinite programs via low-rank factorization. *Mathematical Programming* 95: 329–357.
19. Burer S, Monteiro RDC (2005) Local minima and convergence in low-rank semidefinite programming. *Mathematical Programming* 103: 427–444.
20. Bruzda W, Cappellini V, Sommers H, Życzkowski K (2009) Random quantum operations. *Physics Letters A* 373: 320–324.